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Solvability of a Class of Braided Fusion Categories

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Abstract We show that a weakly integral braided fusion category $\mathcal{C}$ such that every simple object of $\mathcal{C}$ has Frobenius-Perron dimension ≤ 2 is solvable. In addition, we prove that such a fusion category is group-theoretical in the extreme case where the universal grading group of $\mathcal{C}$ is trivial.

Keywords Fusion category · Braided fusion category · Solvability

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1 Introduction and Main Results

Let k be an algebraically closed field of characteristic zero. A fusion category over k is a semisimple rigid tensor category over k having finitely many isomorphism classes of simple objects. In this paper we consider the problem of giving structural results of a fusion category $\mathcal{C}$ under restrictions on the set $\mathrm{c.d.}(\mathcal{C})$ of Frobenius-Perron dimensions of its simple objects.[1]

[1] The notation $\mathrm{c.d.}(\mathcal{C})$ has been chosen by analogy with the notation $\mathrm{c.d.}(G)$ in [12, Chapter 12], where "cd" stands for "character degrees".

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Results of this type were obtained in the paper [20]. For instance, we showed in [20, Theorem 7.3] that under the assumption that $\mathcal{C}$ is braided odd-dimensional and $\mathrm{c.d.}(\mathcal{C}) \subseteq \{p^m : m \geq 0\}$, where p is a (necessarily odd) prime number, then $\mathcal{C}$ is solvable. Also, the same is true when $\mathcal{C} = \operatorname{Rep} H$, where H is a semisimple quasitriangular Hopf algebra and $\mathrm{c.d.}(\mathcal{C}) = \{1, 2\}$ [20, Theorem 6.12].

Using results of the paper [1], we also showed in [20, Theorem 6.4] that if $\mathcal{C} = \operatorname{Rep} H$, where H is any semisimple Hopf algebra, and $\mathrm{c.d.}(\mathcal{C}) \subseteq \{1, 2\}$, then $\mathcal{C}$ is weakly group-theoretical, and furthermore, it is group-theoretical if $\mathcal{C}$ coincides with the adjoint subcategory $\mathcal{C}_{\mathrm{ad}}$.

Our main results are the following theorems. Recall that a fusion category $\mathcal{C}$ is called *weakly integral* if the Frobenius-Perron dimension of $\mathcal{C}$ is a natural integer.

Theorem 1.1 *Let $\mathcal{C}$ be a weakly integral braided fusion category such that* $\operatorname{FPdim} X \leq 2$*, for all simple object X of $\mathcal{C}$. Then $\mathcal{C}$ is solvable.*

Theorem 1.1 extends the previous result for semisimple quasitriangular Hopf algebras mentioned above. It implies in particular that every weakly integral braided fusion category with Frobenius-Perron dimensions of simple objects at most 2 is weakly group-theoretical. This gives some further support to the conjecture that every weakly integral fusion category is weakly group-theoretical. See [8, Question 2].

It is known that a nilpotent braided fusion category, which is in addition integral (that is, $\mathrm{c.d.}(\mathcal{C}) \subseteq \mathbb{Z}_+$) is always group-theoretical [4, Theorem 6.10]. We also show that the same conclusion is true in the opposite extreme case:

Theorem 1.2 *Let $\mathcal{C}$ be a weakly integral braided fusion category such that* $\operatorname{FPdim} X \leq 2$*, for all simple object X of $\mathcal{C}$. Suppose that the universal grading group of $\mathcal{C}$ is trivial. Then $\mathcal{C}$ is group-theoretical.*

Theorems 1.1 and 1.2 are proved in Section 4. Our proofs rely on the results of Naidu and Rowell [18] for the case where $\mathcal{C}$ is integral and has a faithful self-dual simple object of Frobenius-Perron dimension 2.

Being group-theoretical, a braided fusion category $\mathcal{C}$ satisfying the assumptions of Theorem 1.2, has the so called property **F**, namely, all associated braid group representations on the tensor powers of objects of $\mathcal{C}$ factor over finite groups. See [9, Corollary 4.4]. It is conjectured that every braided weakly integral fusion category does have property **F** [18]. This conjecture has been proved for braided fusion categories $\mathcal{C}$ with $\mathrm{c.d.}(\mathcal{C}) = \{1, 2\}$ such that all objects of $\mathcal{C}$ are self-dual or $\mathcal{C}$ is generated by a self-dual simple object [18, Corollary 4.3 and Remark 4.4].

The paper is organized as follows. In Section 2 we recall the main facts and terminology about fusion and braided fusion categories used throughout. In Section 3 we discuss some families of (integral) examples that appear in the literature. We also recall in this section the results of the paper [18] related to dihedral group fusion rules that will be used later. In Section 4 we give the proofs of Theorems 1.1 and 1.2.

2 Preliminaries

2.1 Fusion Categories

Let $\mathcal{C}$ be a fusion category. We shall denote by $\mathrm{Irr}(\mathcal{C})$ the set of isomorphism classes of simple objects of $\mathcal{C}$ and by $G(\mathcal{C})$ the group of isomorphism classes of invertible objects of $\mathcal{C}$. For an object X of $\mathcal{C}$, we shall indicate by $\mathcal{C}[X]$ the fusion subcategory generated by X and by $G[X]$ the subgroup of $G(\mathcal{C})$ consisting of invertible objects g such that $g \otimes X \simeq X$.

If $\mathcal{D}$ is another fusion category, $\mathcal{C}$ and $\mathcal{D}$ are *Morita equivalent* if $\mathcal{D}$ is equivalent to the dual $\mathcal{C}^*_{\mathcal{M}}$ with respect to an indecomposable module category $\mathcal{M}$. Recall that $\mathcal{C}$ is called *pointed* if all its simple objects are invertible and it is called *group-theoretical* if it is Morita equivalent to a pointed fusion category.

There is a canonical faithful grading $\mathcal{C} = \oplus_{g\in U(\mathcal{C})}\mathcal{C}_g$, with trivial component $\mathcal{C}_e = \mathcal{C}_{\mathrm{ad}}$, where $\mathcal{C}_{\mathrm{ad}}$ is the *adjoint subcategory* of $\mathcal{C}$, that is, the fusion subcategory generated by $X \otimes X^*$, where X runs through the simple objects of $\mathcal{C}$. The group $U(\mathcal{C})$ is called the *universal grading group* of $\mathcal{C}$. $\mathcal{C}$ is called nilpotent if the upper central series $\cdots \subseteq \mathcal{C}^{(n+1)} \subseteq \mathcal{C}^{(n)} \subseteq \cdots \subseteq \mathcal{C}^{(0)} = \mathcal{C}$ converges to Vec_k, where $\mathcal{C}^{(i)} := (\mathcal{C}^{(i-1)})_{\mathrm{ad}}$, $i \geq 1$. See [11].

A *weakly group-theoretical* fusion category is a fusion category $\mathcal{C}$ which is Morita equivalent to a nilpotent fusion category. If $\mathcal{C}$ is Morita equivalent to a cyclically nilpotent fusion category, then $\mathcal{C}$ is called *solvable*. We refer the reader to [7, 8] for further definitions and facts about fusion categories.

2.2 Braided Fusion Categories

Let $\mathcal{C}$ be a *braided* fusion category, that is, $\mathcal{C}$ is equipped with natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$, $X, Y \in \mathcal{C}$, satisfying the hexagon axioms. Recall that $\mathcal{C}$ is called *premodular* if it is also spherical, that is, $\mathcal{C}$ has a pivotal structure such that left and right categorical dimensions coincide. Equivalently, $\mathcal{C}$ is premodular if it is endowed with a compatible ribbon structure [2, 16].

We say that the objects X and Y of a braided fusion category $\mathcal{C}$ centralize each other if $c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}$. The *centralizer* $\mathcal{D}'$ of a fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$ is defined to be the full subcategory of objects of $\mathcal{C}$ that centralize every object of $\mathcal{D}$. The centralizer $\mathcal{D}'$ results in a fusion subcategory of $\mathcal{C}$.

The *Müger (or symmetric) center* $Z_2(\mathcal{C})$ of $\mathcal{C}$ is $Z_2(\mathcal{C}) = \mathcal{C}'$; this is a symmetric fusion subcategory of $\mathcal{C}$ whose objects are called central, degenerate or transparent. A braided fusion category $\mathcal{C}$ is called *non-degenerate* if its Müger center $Z_2(\mathcal{C})$ is trivial. A *modular* category is a non-degenerate premodular category $\mathcal{C}$.

Remark 2.1 Recall that a fusion category $\mathcal{C}$ is called pseudo-unitary if $\dim \mathcal{C} = \mathrm{FPdim}\, \mathcal{C}$, where $\dim \mathcal{C}$ is the global dimension of $\mathcal{C}$ and $\mathrm{FPdim}\, \mathcal{C}$ is the Frobenius-Perron dimension of $\mathcal{C}$. If $\mathcal{C}$ is pseudo-unitary then $\mathcal{C}$ has a canonical spherical structure with respect to which categorical dimensions of all simple objects coincide with their Frobenius-Perron dimensions [7, Proposition 8.23].

In particular, this holds for any weakly integral fusion category, because it is automatically pseudo-unitary [7, Proposition 8.24]. Hence every weakly integral non-degenerate fusion category is canonically a modular category.

3 Some Families of Examples

3.1 Examples of Fusion Categories with Frobenius-Perron Dimensions ≤ 2

In this subsection we discuss examples of weakly integral fusion categories with Frobenius-Perron dimensions of simple objects ≤ 2 that appear in the literature.

Example 3.1 Consider a Hopf algebra H fitting into an abelian exact sequence:

$$k \to k^{\Gamma} \to H \to k\mathbb{Z}_2 \to k, \tag{3.1}$$

where Γ is a finite group. Let $\mathcal{C} = \operatorname{Rep} H$. Then $\operatorname{c.d.}(\mathcal{C}) \subseteq \{1, 2\}$ and equality holds if the associated action of $\mathbb{Z}_2$ on Γ is not trivial.

All these examples are group-theoretical, in view of [19, Theorem 1.3]. Observe that, as a consequence of [1, Theorem 6.4], any cosemisimple Hopf algebra H such that $\operatorname{c.d.}(\mathcal{C}) \subseteq \{1, 2\}$ is group-theoretical if $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$. See [20, Theorem 6.4].

Non-trivial examples of cosemisimple Hopf algebras fitting into an exact sequence (3.1) are given by the Hopf algebras

$$\mathcal{A}_{4m}^*, \mathcal{B}_{4m}^* \quad m \geq 2,$$

of dimension $4m$, due to Masuoka [14]. In these cases, Γ is a dihedral group.

Example 3.2 Let $\mathcal{C} = \mathcal{TY}(G, \chi, \tau)$ be the Tambara-Yamagami category associated to a finite (necessarily abelian) group G, a symmetric non-degenerate bicharacter $\chi : G \times G \to k^{\times}$ and an element $\tau \in k$ satisfying $|G|\tau^2 = 1$ [24]. This is a fusion category with isomorphism classes of simple objects parameterized by the set $G \cup \{X\}$, where $X \notin G$, obeying the fusion rules

$$g \otimes h = gh, \quad g, h \in G, \quad X \otimes X = \oplus_{g \in G} g. \tag{3.2}$$

We have $\operatorname{c.d.}(\mathcal{C}) = \{1, 2\}$ if and only if G is of order 4. Therefore, in this case $\operatorname{FPdim} \mathcal{C} = 8$.

If $G \simeq \mathbb{Z}_4$, there are two possible fusion categories $\mathcal{C}$. None of them is braided [22, Theorem 1.2 (1)].

If $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ there are exactly four classes of Tambara-Yamagami categories with irreducibles degrees 1 or 2, by [24, Theorem 4.1]. Three of them are (equivalent to) the categories of representations of eight-dimensional Hopf algebras: the dihedral group algebra of order 8, the quaternion group algebra, and the Kac-Paljutkin Hopf algebra H_8. The remaining fusion category, which has the same χ as H_8 but $\tau = -1/2$, is not realized as the fusion category of representations of a Hopf algebra. Since in this case G is an elementary abelian 2-group all of these categories admit a braiding, by [22, Theorem 1.2 (1)].

All the fusion categories in this example are group-theoretical. In fact, by [10, Lemma 4.5], for any symmetric non-degenerate bicharacter $\chi : G \times G \to k^{\times}$, G

contains a Lagrangian subgroup with respect to χ. Therefore $\mathcal{TY}(G, \chi, \tau)$ is group-theoretical, by [10, Theorem 4.6].

Example 3.3 Recall that a near-group category is a fusion category with exactly one isomorphism class of non-invertible simple object. In the notation of [22], the fusion rules of $\mathcal{C}$ are determined by a pair (G, κ), where G is the group of invertible objects of $\mathcal{C}$ and κ is a nonnegative integer. Letting $\operatorname{Irr}(\mathcal{C}) = G \cup \{X\}$, where X is non-invertible, we have the relation

$$X \otimes X = \oplus_{g \in G} g \oplus \kappa X. \tag{3.3}$$

Near-group categories with fusion rule $(G, 0)$ for some finite group G are thus Tambara-Yamagami categories, discussed in the previous example. Let us consider near-group categories with fusion rule (G, κ) for some finite group G and a positive integer κ.

We have $\mathrm{c.d.}(\mathcal{C}) = \{1, 2\}$ if and only if G is of order 2 and $\kappa = 1$, that means $\mathcal{C}$ is of type $(\mathbb{Z}_2, 1)$. Therefore, in this case $\operatorname{FPdim} \mathcal{C} = 6$ and since $\kappa > 0$, then $\mathcal{C}$ is group-theoretical, by [6, Theorem 1.1]. By [25, Theorem 1.5], there are up to equivalence exactly two non-symmetric braided near-group categories with fusion rule $(\mathbb{Z}_2, 1)$.

Example 3.4 Examples of weakly integral braided fusion categories which are not integral and Frobenius-Perron dimensions of simple objects are ≤ 2 are given by the Ising categories, studied in [5, Appendix B]. In this case, there is a unique non-invertible simple object X with $X^{\otimes 2} = \mathbf{1} \oplus a$, where a generates the group of invertible objects, isomorphic to $\mathbb{Z}_2$ (note that these are also Tambara-Yamagami categories). We have here $\mathrm{c.d.}(\mathcal{C}) = \{1, \sqrt{2}\}$ and $\operatorname{FPdim} \mathcal{C} = 4$. Every braided Ising category is modular [5, Corollary B.12].

Other examples come from braided fusion categories with generalized Tambara-Yamagami fusion rules of type $(G, \mathbb{Z}_2)$, where G is a finite group. See [13]. In these examples, $\mathcal{C}$ is not pointed, the group of invertible objects is G, and $\mathbb{Z}_2 \simeq \Gamma \subseteq G$ is a subgroup such that $X \otimes X^* \simeq \oplus_{h \in \Gamma} h$, for all non-invertible object X of $\mathcal{C}$. Hence we also have $\mathrm{c.d.}(\mathcal{C}) = \{1, \sqrt{2}\}$.

Since they are not integral, these examples are not group-theoretical.

Example 3.5 Let $\mathcal{C}$ be a braided group-theoretical fusion category. Then $\mathcal{C}$ is an equivariantization of a pointed fusion category, that is, $\mathcal{C} \simeq \mathcal{D}^G$, where $\mathcal{D}$ is a pointed fusion category and G is a finite group acting on $\mathcal{D}$ by tensor autoequivalences [17]. In this case, $\mathcal{C}$ contains the category $\operatorname{Rep} G$ of finite-dimensional representations of G as a fusion subcategory.

Suppose that $\mathrm{c.d.}(\mathcal{C}) = \{1, p\}$, where p is any prime number. Then also $\mathrm{c.d.}(G) \subseteq \{1, p\}$. In particular, the group G must have a normal abelian p-complement; moreover, either G contains an abelian normal subgroup of index p or the center $Z(G)$ has index p^3. See [12, Theorems 6.9, 12.11].

3.2 Fusion Rules of Dihedral Type

Let D_n be the dihedral group of order $2n$, $n \geq 1$. Recall that D_n has a presentation by generators t, z and relations $t^2 = 1 = z^n$, $tz = z^{-1}t$.

The following proposition describes the fusion rules of $\operatorname{Rep} D_n$ (*c.f.* [14]).

Proposition 3.6

(1) *Suppose n is odd. Then the isomorphisms classes of simple objects of* $\operatorname{Rep} D_n$ *are represented by 2 invertible objects, $\mathbf{1}$ and g, and $r = (n-1)/2$ simple objects $X_1, \dots, X_r$, of dimension 2, such that*

$$g \otimes X_i = X_i = X_i \otimes g, \qquad \forall i = 1, \dots, r,$$

$$X_i \otimes X_j = \begin{cases} X_{i+j} \oplus X_{|i-j|}, & \text{if } \ i+j \leq r, \\ X_{n-(i+j)} \oplus X_{|i-j|}, & \text{if } \ i+j > r; \end{cases}$$

where $X_0 = \mathbf{1} \oplus g$.

(2) *Suppose n is even, that is $n = 2m$. Then the isomorphisms classes of simple objects of* $\operatorname{Rep} D_n$ *are represented by 4 invertible objects, $\mathbf{1}$, g, h, $f = gh$, and $m-1$ simple objects $X_1, \dots, X_{m-1}$, of dimension 2, such that*

$$g \otimes X_i = X_i = X_i \otimes g, \qquad \forall i = 1, \dots, m-1,$$

$$h \otimes X_i = X_{m-i} = X_i \otimes h, \qquad \forall i = 1, \dots, m-1,$$

$$X_i \otimes X_j = \begin{cases} X_{i+j} \oplus X_{|i-j|}, & \text{if } \ i+j \leq m, \\ X_{2m-(i+j)} \oplus X_{|i+j|}, & \text{if } \ i+j > m; \end{cases}$$

where $X_0 = \mathbf{1} \oplus g$ and $X_m = h \oplus f$.

In particular, the group of invertible objects in $\operatorname{Rep} D_n$ *is isomorphic to $\mathbb{Z}_2$ if n is odd, and to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even.*

Remark 3.7 Suppose that 4 divides $n = 2m$. Then $X_{m/2}$ is fixed under (left and right) multiplication by all invertible objects of $\operatorname{Rep} D_n$.

Let $\mathcal{C}$ be a fusion category with $\operatorname{c.d.}(\mathcal{C}) = \{1, 2\}$. Suppose that the Grothendieck ring of $\mathcal{C}$ is commutative (for example, this is the case if $\mathcal{C}$ is braided). Assume in addition that the following conditions hold:

(a) All objects are self-dual, that is $X \simeq X^*$, for every object X of $\mathcal{C}$.
(b) $\mathcal{C}$ has a faithful simple object.

Then, it is shown in [18, Theorem 4.2] that $\mathcal{C}$ is Grothendieck equivalent to $\operatorname{Rep} D_n$. Moreover, $\mathcal{C}$ is necessarily group-theoretical.

It is possible to remove the assumption that all the objects are self-dual but it is still necessary to keep the condition of self-duality on the faithful simple object. Namely, suppose that $\mathcal{C}$ is not self-dual, but satisfies

(b') $\mathcal{C}$ has a faithful self-dual simple object.

In this case $\mathcal{C}$ is still group-theoretical and it is Grothendieck equivalent to $\operatorname{Rep} \widetilde{D}_n$, n odd. See [18, Remark 4.4]. Here $\widetilde{D}_n$ is the generalized quaternion (binary dihedral) group of order $4n$, that is, the group presented by generators a, s, with relations $a^{2n} = 1$, $s^2 = a^n$, $s^{-1}as = a^{-1}$. (Observe that for n odd, $\widetilde{D}_n$ is isomorphic to the semidirect product $\mathbb{Z}_n \rtimes \mathbb{Z}_4$, with respect to the action given by inversion, considered in [18]. For even n, Rep $\widetilde{D}_n$ is Grothendieck equivalent to Rep D_{2n}, while $\mathbb{Z}_n \rtimes \mathbb{Z}_4$ has no faithful representation of degree 2.)

Lemma 3.8 *Let $n \geq 2$. Then* $(\text{Rep } \widetilde{D}_n)_{\text{ad}} = \text{Rep } D_n$*. In addition,*

$$(\text{Rep } D_n)_{\text{ad}} = \begin{cases} \text{Rep } D_{n/2}, & \textit{if} \quad n \quad \textit{is even}, \\ \text{Rep } D_n, & \textit{if} \quad n \quad \textit{is odd}. \end{cases}$$

Proof Recall that when $\mathcal{C} = \text{Rep}\, G$, where G is a finite group, then $\mathcal{C}_{\text{ad}} = \text{Rep}\, G/Z(G)$ [11]. The first claim follows from the fact that the center of $\tilde{D}_n$ equals $\{1, s^2\} \simeq \mathbb{Z}_2$. On the other hand, the center $Z(D_n)$ is trivial if n is odd, and equals $\{1, z^{n/2}\} \simeq \mathbb{Z}_2$ if n is even. This implies the second claim and finishes the proof of the lemma. □

4 Proof of the Main Results

In this section we shall prove Theorems 1.1 and 1.2.

Proposition 4.1 *Let $\mathcal{C}$ be a premodular fusion category. Suppose $\mathcal{C}$ has an invertible object g of order n and a simple object X such that*

$$g \otimes X = X, \text{ and} \tag{4.1}$$

$$g \text{ centralizes } X. \tag{4.2}$$

Then we have

(i) *$\mathcal{C}$ is an equivariantization by the cyclic group $\mathbb{Z}_n$ of a fusion category $\widetilde{\mathcal{C}}$.*
(ii) *If $g \in \mathcal{C}'$, then $\widetilde{\mathcal{C}}$ is braided.*

Proof Condition 4.1 ensures the existence of a fiber functor on the fusion category $\mathcal{C}[g]$ generated by g. Then $\mathcal{C}[g]$ is equivalent to Rep $\mathbb{Z}_n$ as fusion categories.

Moreover, they are equivalent as braided fusion categories. Indeed, Eq. 4.1 implies $\mathcal{C}[g] \subseteq \mathcal{C}[X]$ and therefore $\mathcal{C}[g] \subseteq Z_2(\mathcal{C}[X])$, by Eq. 4.2. Hence $\mathcal{C}[g]$ is symmetric. Then the only possible twists in $\mathcal{C}$ are $\theta_h = 1$ and $\theta_h = -1$ for all $h \in \langle g \rangle$. But θ_h is not equal to -1 since h centralizes X and $h \otimes X = X$ [15, Lemma 5.4]. Then $\theta_h = 1$ for all $h \in \langle g \rangle$. Therefore $\mathcal{C}[g] \simeq \text{Rep}\, \mathbb{Z}_n$ as braided fusion categories, as claimed.

Let $\Gamma = \langle g \rangle \subseteq G(\mathcal{C})$. It follows from [5, Theorem 4.18 (i)] that the de-equivariantization $\widetilde{\mathcal{C}} = \mathcal{C}_\Gamma$ of $\mathcal{C}$ by Γ is a fusion category and there is a canonical equivalence $\mathcal{C} \simeq \widetilde{\mathcal{C}}^\Gamma$ between the category $\mathcal{C}$ and the Γ-equivariantization of $\widetilde{\mathcal{C}}$, which shows (i).

Furthermore, if $g \in \mathcal{C}'$ then $\widetilde{\mathcal{C}}$ is braided and the equivalence $\mathcal{C} \simeq \widetilde{\mathcal{C}}^\Gamma$ is of braided fusion categories [2, 15] (see also [5, Theorem 4.18 (ii)]). Thus we get (ii). This proves the proposition. □

Lemma 4.2 *Let $\mathcal{C}$ be a fusion category with commutative Grothendieck ring. Suppose that $\mathcal{C} = \mathcal{C}_{\text{ad}}$. If $\mathcal{D}_1, \ldots, \mathcal{D}_s$ are fusion subcategories that generate $\mathcal{C}$ as a fusion category, then $\mathcal{D}_1^{(m)}, \ldots, \mathcal{D}_s^{(m)}$ generate $\mathcal{C}$ as a fusion category, $\forall m \geq 0$.*

Proof Since $\mathcal{D}_1, \ldots, \mathcal{D}_s$ generate $\mathcal{C}$, then $(\mathcal{D}_1)_{\mathrm{ad}}, \ldots, (\mathcal{D}_s)_{\mathrm{ad}}$ generate $\mathcal{C}$. In fact, let X be a simple object of $\mathcal{C}$. There exist simple objects $X_{i_1}, \ldots, X_{i_t}$, with $X_{i_l} \in \mathcal{D}_{i_l}$, $1 \leq i_1, \ldots, i_t \leq s$, such that X is a direct summand of $X_{i_1} \otimes \ldots \otimes X_{i_t}$. Then $X \otimes X^*$ is a direct summand of

$$X_{i_1} \otimes \ldots \otimes X_{i_t} \otimes X_{i_t}^* \otimes \ldots \otimes X_{i_1}^* \simeq (X_{i_1} \otimes X_{i_1}^*) \otimes \ldots \otimes (X_{i_t} \otimes X_{i_t}^*),$$

where we have used that $\mathcal{C}$ has a commutative Grothendieck ring. Notice that the object in the right hand side belongs to the fusion subcategory generated by $(\mathcal{D}_1)_{\mathrm{ad}}, \ldots, (\mathcal{D}_s)_{\mathrm{ad}}$. Since X was arbitrary, it follows that $(\mathcal{D}_1)_{\mathrm{ad}}, \ldots, (\mathcal{D}_s)_{\mathrm{ad}}$ generate $\mathcal{C}_{\mathrm{ad}}$. But $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$ by assumption, then we have proved that $(\mathcal{D}_1)_{\mathrm{ad}}, \ldots, (\mathcal{D}_s)_{\mathrm{ad}}$ generate $\mathcal{C}$. The statement follows from this by induction on n, since $\mathcal{D}_j^{(n)} = (\mathcal{D}_j^{(n-1)})_{\mathrm{ad}}$, for all $j = 1, \ldots s, n \geq 1$. □

4.1 Braided Fusion Categories with Irreducible Degrees 1 and 2

Throughout this subsection $\mathcal{C}$ is a braided fusion category with $\mathrm{c.d.}(\mathcal{C}) = \{1, 2\}$. We regard $\mathcal{C}$ as a premodular category with respect to its canonical spherical structure. See Remark 2.1.

Remark 4.3 Note that $G[X] \neq \mathbf{1}$, for all X such that $\operatorname{FPdim} X = 2$. Moreover, $|G[X]| = 2$ or 4. In particular the (abelian) group $G(\mathcal{C})$ is not trivial.

Proposition 4.4 *Let g be a non-trivial invertible object such that $g^2 = 1$ and $\theta_g = 1$. Assume that g generates the Müger center $\mathcal{C}'$ of $\mathcal{C}$ as a fusion category. Then $\mathcal{C}$ is the equivariantization of a modular fusion category $\widetilde{\mathcal{C}}$ by the group $\mathbb{Z}_2$. Furthermore* $\mathrm{c.d.}(\widetilde{\mathcal{C}}) \subseteq \{1, 2\}$.

Proof By assumption $\mathcal{C}' \simeq \operatorname{Rep} \mathbb{Z}_2$ is tannakian. Then the de-equivariantization $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ by $\mathcal{C}'$ is a modular category and there is an action of $\mathbb{Z}_2$ on $\widetilde{\mathcal{C}}$ such that $\mathcal{C} \simeq \widetilde{\mathcal{C}}^{\mathbb{Z}_2}$ [2, 15]. Since $\mathrm{c.d.}(\widetilde{\mathcal{C}}^{\mathbb{Z}_2}) = \mathrm{c.d.}(\mathcal{C}) = \{1, 2\}$, then $\mathrm{c.d.}(\widetilde{\mathcal{C}}) \subseteq \{1, 2\}$, by [8, Proof of Proposition 6.2], [20, Lemma 7.2]. □

Lemma 4.5 *Suppose that $\mathcal{C} \neq \mathcal{C}_{\mathrm{ad}}$ and $\mathcal{C}_{\mathrm{ad}}$ is solvable. Then $\mathcal{C}$ is solvable.*

Proof Since $\mathcal{C}$ is braided, its universal grading group $U(\mathcal{C})$ is abelian [11, Theorem 6.2]. The category $\mathcal{C}$ is a $U(\mathcal{C})$-extension of $\mathcal{C}_{\mathrm{ad}}$ and an extension of a solvable category by a solvable group is again solvable [8, Proposition 4.5 (i)]. Then $\mathcal{C}$ is solvable, as claimed. □

Lemma 4.6 *Assume $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$. Then* $\operatorname{FPdim} \mathcal{C}' \geq 2$.

Proof Suppose on the contrary that $\operatorname{FPdim} \mathcal{C}' = 1$, that is, $\mathcal{C}$ is modular. Then, by [11, Theorem 6.2], $U(\mathcal{C}) \simeq \widehat{G(\mathcal{C})} \simeq G(\mathcal{C})$. By Remark 4.3, $\mathcal{C}_{\mathrm{ad}} \subsetneq \mathcal{C}$, against the assumption. Hence $\operatorname{FPdim} \mathcal{C}' \geq 2$, as claimed. □

Lemma 4.7 *Suppose $\mathcal{C}$ is generated by a simple object X such that $X \simeq X^*$ and* $\mathrm{FPdim}\, X = 2$. *Then we have*

(i) *$\mathcal{C}$ is not modular.*

Assume $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$. Then we have in addition

(ii) *There is a group isomorphism $G(\mathcal{C}) \simeq \mathbb{Z}_2$.*
(iii) *$G(\mathcal{C}) \subseteq \mathcal{C}'$.*

Proof Since $\mathrm{c.d.}(\mathcal{C}) = \{1, 2\}$, then by [18, Theorem 4.2; Remark 4.4], $\mathcal{C}$ is Grothendieck equivalent to $\mathrm{Rep}\, D_n$ or $\mathrm{Rep}\, \widetilde{D}_{2n+1}$, for some $n \geq 1$. Since the universal grading group is a Grothendieck invariant, then in the first case $U(\mathcal{C})$ is isomorphic to $\mathbb{Z}_2$ if n is even and is trivial if n is odd. But $G(\mathcal{C})$, which is also a Grothendieck invariant, is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even and is isomorphic to $\mathbb{Z}_2$ if n is odd, by Proposition 3.2. Then $U(\mathcal{C})$ is not isomorphic to $\widehat{G(\mathcal{C})}$, for any n. Therefore $\mathcal{C}$ is not modular, by [11, Theorem 6.2]. Similarly, if $\mathcal{C}$ is Grothendieck equivalent to $\mathrm{Rep}\, \widetilde{D}_{2n+1}$, we have $U(\mathcal{C}) \simeq \mathbb{Z}_2$ and $G(\mathcal{C}) \simeq \mathbb{Z}_4$. Hence $\mathcal{C}$ is not modular in this case neither. This shows (i).

Notice that the assumption $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$ implies that $\mathcal{C}$ is Grothendieck equivalent to $\mathrm{Rep}\, D_n$, for some n odd. Then (ii) follows immediately from the fusion rules of $\mathrm{Rep}\, D_n$, with n odd (see Proposition 3.2). Since, by (i), $\mathcal{C}'$ is not trivial, then $G(\mathcal{C}') \neq \mathbf{1}$, because $\mathrm{c.d.}(\mathcal{C}') \subseteq \{1, 2\}$ (*c.f.* Remark 4.3). By part (ii), $G(\mathcal{C}') = G(\mathcal{C})$ and (iii) follows. □

Remark 4.8 If $\mathcal{C}$ is a fusion category as in Lemma 4.7, then the assumption $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$ is equivalent to saying that $\mathcal{C}$ is Grothendieck equivalent to $\mathrm{Rep}\, D_n$, for some $n \geq 1$ *odd*.

Lemma 4.9 *Suppose that $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$. Then $\mathcal{C}$ is generated by fusion subcategories $\mathcal{D}_1, \dots, \mathcal{D}_s$, $s \geq 1$, where $\mathcal{D}_i$ is Grothendieck equivalent to* $\mathrm{Rep}\, D_{n_i}$ *and n_i is an odd natural number, for all $i = 1, \dots, s$.*

Proof Let $\mathcal{C} = \mathcal{C}[X_1, \dots, X_s]$ for some simple objects $X_1, \dots, X_s$. Let $\mathcal{D}_i = \mathcal{C}[X_i]$ be the fusion subcategory generated by X_i, $i = 1, \dots, s$. By Lemma 4.2, $(\mathcal{D}_1)_{\mathrm{ad}}, \dots, (\mathcal{D}_s)_{\mathrm{ad}}$ generate $\mathcal{C}$ as a fusion category. Hence, it is enough to consider only those simple objects X_i whose Frobenius-Perron dimension equals 2 (otherwise, $\mathrm{FPdim}\, X_i = 1$ and $X_i \otimes X_i^* \simeq \mathbf{1}$).

Moreover, iterating the application of Lemma 4.2, we may further assume that $|G[X_i]| = 2$, for all $i = 1, \dots, s$. Thus we have a decomposition $X_i \otimes X_i^* \simeq \mathbf{1} \oplus g_i \oplus X_i'$, where $G[X_i] = \{\mathbf{1}, g_i\}$ and X_i' is a self-dual simple object of Frobenius-Perron dimension 2. Since $X_i \otimes X_i^*$ generates $(\mathcal{D}_i)_{\mathrm{ad}}$, the above reductions allow us to assume that $\mathcal{D}_i = \mathcal{C}[X_i]$ with X_i simple objects of $\mathcal{C}$ such that $\mathrm{FPdim}\, X_i = 2$ and $X_i \simeq X_i^*$, $\forall i = 1, \dots, s$.

By [18, Theorem 4.2; Remark 4.4], $\mathcal{D}_i$ is Grothendieck equivalent to $\mathrm{Rep}\, D_{n_i}$ or to $\mathrm{Rep}\, \widetilde{D}_{2n_i+1}$. Iterating the application of Lemma 4.2 and using Lemma 3.8, we obtain that $\mathcal{C} = \mathcal{C}[\mathcal{D}_1, \dots, \mathcal{D}_s]$, with $\mathcal{D}_j$ a fusion subcategory of $\mathcal{C}$ Grothendieck equivalent to $\mathrm{Rep}\, D_{n_j}$, n_j odd, for all $j = 1, \dots, s$, as we wanted. □

4.2 Proof of Theorems 1.1 and 1.2

Let $\mathcal{C}$ be a weakly integral fusion category. It follows from [11, Theorem 3.10] that either $\mathcal{C}$ is integral, or $\mathcal{C}$ is a $\mathbb{Z}_2$-extension of a fusion subcategory $\mathcal{D}$. In particular, if $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$, then $\mathcal{C}$ is necessarily integral.

Lemma 4.10 *Let $\mathcal{C}$ be fusion category and let X, X' be simple objects of $\mathcal{C}$. Then the following are equivalent:*

(i) *The tensor product $X^* \otimes X'$ is simple.*
(ii) *For every simple object $Y \neq \mathbf{1}$ of $\mathcal{C}$, either $m(Y, X \otimes X^*) = 0$ or $m(Y, X' \otimes X'^*) = 0$.*

In particular, if $X^* \otimes X'$ is not simple, then $\mathcal{C}[X]_{\mathrm{ad}} \cap \mathcal{C}[X']_{\mathrm{ad}}$ is not trivial.

Proof The equivalence between (i) and (ii) is proved in [1, Lemma 6.1] in the case where $\mathcal{C}$ is the category of (co)representations of a semisimple Hopf algebra. Note that the proof *loc. cit.* works in this more general context as well. □

Proof of Theorem 1.1 The proof is by induction on $\operatorname{FPdim} \mathcal{C}$. As pointed out at the beginning of this subsection, if $\mathcal{C}$ is not integral, then it is a $\mathbb{Z}_2$-extension of a fusion subcategory $\mathcal{D}$. Since $\mathcal{D}$ also satisfies the assumptions of the theorem, then $\mathcal{D}$ is solvable, by induction. Hence $\mathcal{C}$ is solvable as well.

We may thus assume that $\mathcal{C}$ is integral. Therefore $\mathrm{c.d.}(\mathcal{C}) = \{1, 2\}$ and the results of the previous subsection apply. By Lemma 4.5, we may assume that $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$. Then it follows from Lemma 4.9 that $\mathcal{C} = \mathcal{C}[\mathcal{D}_1, \dots, \mathcal{D}_s]$, with $\mathcal{D}_j$ Grothendieck equivalent to $\operatorname{Rep} D_{n_j}$, n_j odd, $\forall j = 1, \dots, s$.

By Lemma 4.7, $G(\mathcal{D}_j) = \{\mathbf{1}, g_j\}$, $\forall j = 1, \dots, s$. We claim that $g_i = g_j$ $\forall 1 \leq i, j \leq s$. Indeed, let $\mathcal{D}_j = \mathcal{C}[X^{(j)}]$, where $X^{(j)} = X_1^{(j)}$ in the notation of Proposition 3.6. Then we have $(X^{(j)})^{\otimes 2} = \mathbf{1} \oplus g_j \oplus X_2^{(j)}$. Fix $1 \leq i, j \leq s$. Since $\mathcal{C}$ has no simple objects of Frobenius-Perron dimension 4 then $g_i = g_j$ or $X_2^{(j)} \simeq X_2^{(i)}$, by Lemma 4.10. In the first case we are done. In the second case, we note that $\{\mathbf{1}, g_j\} = G[X_2^{(j)}] = G[X_2^{(i)}] = \{\mathbf{1}, g_i\}$. Then $g_j = g_i$, as claimed. Let $g = g_j = g_i$.

By Lemma 4.7, $g \in \mathcal{D}_i'$, for all $i = 1, \dots, s$. Since $\mathcal{D}_i$, $1 \leq i \leq s$, generate $\mathcal{C}$ then $g \in \mathcal{C}'$. It follows from Theorem 4.1 (ii) that $\mathcal{C}$ is the equivariantization by $\mathbb{Z}_2$ of a braided fusion category $\widetilde{\mathcal{C}}$. In particular, $\operatorname{FPdim} \widetilde{\mathcal{C}} = \operatorname{FPdim} \mathcal{C}/2$ and $\mathrm{c.d.}(\widetilde{\mathcal{C}}) \subseteq \{1, 2\}$, by [8, Proof of Proposition 6.2 (1)], [20, Lemma 7.2]. By inductive hypothesis, $\widetilde{\mathcal{C}}$ is solvable. Then $\mathcal{C}$, being the equivariantization of a solvable fusion category by a solvable group is itself solvable [8, Proposition 4.5 (i)]. □

Theorem 4.11 *Let $\mathcal{C}$ be a weakly integral braided fusion category such that* $\operatorname{FPdim} X \leq 2$ *for all simple object X of $\mathcal{C}$. Assume in addition that $\mathcal{C} = \mathcal{C}_{\mathrm{ad}}$. Then $\mathcal{C}$ is tensor Morita equivalent to a pointed fusion category $\mathcal{C}(A \rtimes \mathbb{Z}_2, \tilde{\omega})$, where A is an abelian group endowed with an action of $\mathbb{Z}_2$ by group automorphisms, and $\tilde{\omega}$ is a certain 3-cocycle on the semidirect product $A \rtimes \mathbb{Z}_2$.*

Proof The assumption $\mathcal{C} = \mathcal{C}_{ad}$ implies that $\mathcal{C}$ is integral. Hence we may assume that $\mathrm{c.d.}(\mathcal{C}) = \{1, 2\}$. By Lemma 4.9, $\mathcal{C}$ is generated by fusion subcategories $\mathcal{D}_1, \dots, \mathcal{D}_s$, $s \geq 1$, where $\mathcal{D}_i$ is Grothendieck equivalent to $\operatorname{Rep} D_{n_i}$ and n_i is an odd natural number, for all $i = 1, \dots, s$. Furthermore, as in the proof of Theorem 1.1, the assumption that $\mathcal{C} = \mathcal{C}_{ad}$ implies that $G(\mathcal{D}_i) = \{\mathbf{1}, g\}$, for all $1 \leq i \leq s$, and $\mathcal{C}[g] \simeq \operatorname{Rep} \mathbb{Z}_2$ is a tannakian subcategory of the Müger center $\mathcal{C}'$. Thus $\mathcal{C} \simeq \tilde{\mathcal{C}}^{\mathbb{Z}_2}$ is an equivariantization of a braided fusion category $\tilde{\mathcal{C}}$.

Equivariantization under a group action gives rise to exact sequences of fusion categories [3, Section 5.3]. In our situation we have an exact sequence of braided tensor functors

$$\operatorname{Rep} \mathbb{Z}_2 \to \mathcal{C} \xrightarrow{F} \tilde{\mathcal{C}}. \tag{4.3}$$

In addition, since $\mathcal{C}[g] \subseteq \mathcal{D}_i$, then Eq. 4.3 induces by restriction an exact sequence

$$\operatorname{Rep} \mathbb{Z}_2 \to \mathcal{D}_i \to \tilde{\mathcal{C}}_i, \tag{4.4}$$

for all $i = 1, \dots, s$, where $\tilde{\mathcal{C}}_i$ is the essential image of $\mathcal{D}_i$ in $\tilde{\mathcal{C}}$ under the functor F. Hence $\tilde{\mathcal{C}}_i$ is a fusion subcategory of $\tilde{\mathcal{C}}$, for all i, and moreover $\tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_s$ generate $\tilde{\mathcal{C}}$ as a fusion category. Note in addition that $\mathrm{c.d.}(\tilde{\mathcal{C}}), \mathrm{c.d.}(\tilde{\mathcal{C}}_i) \subseteq \{1, 2\}$, for all $i = 1, \dots, s$. On the other hand, exactness of the sequence (Eq. 4.4) implies that $2n_i = \operatorname{FPdim} \mathcal{D}_i = 2 \operatorname{FPdim} \tilde{\mathcal{C}}_i$ [3, Proposition 4.10]. Hence $\operatorname{FPdim} \tilde{\mathcal{C}}_i = n_i$ is an odd natural number.

Since $\tilde{\mathcal{C}}_i$ is an integral braided fusion category, then the Frobenius-Perron dimension of every simple object of $\tilde{\mathcal{C}}_i$ divides the Frobenius-Perron dimension of $\tilde{\mathcal{C}}_i$ [8, Theorem 2.11]. Thus we get that $\operatorname{FPdim} Y = 1$, for all $Y \in \operatorname{Irr}(\tilde{\mathcal{C}}_i)$. That is, $\tilde{\mathcal{C}}_i$ is a pointed braided fusion category, for all $i = 1, \dots, s$. Since $\tilde{\mathcal{C}}_1, \dots, \tilde{\mathcal{C}}_s$ generate $\tilde{\mathcal{C}}$ as a fusion category, then $\tilde{\mathcal{C}}$ is also pointed. Therefore $\tilde{\mathcal{C}} \simeq \mathcal{C}(A, \omega)$ as fusion categories, where A is an abelian group and $\omega \in H^3(A, k^\times)$.

Group actions on pointed categories were classified by Tambara [23]. In view of [23, Theorem 4.1] and [21, Proposition 3.2], the fusion category $\mathcal{C} \simeq \tilde{\mathcal{C}}^{\mathbb{Z}_2}$ is tensor Morita equivalent to a pointed category $\mathcal{C}(A \rtimes \mathbb{Z}_2, \tilde{\omega})$, where the semidirect product $A \rtimes \mathbb{Z}_2$ is with respect of the induced action of $\mathbb{Z}_2$ on the group A of invertible objects of $\tilde{\mathcal{C}}$, and $\tilde{\omega}$ is a certain 3-cocycle on $A \rtimes \mathbb{Z}_2$. □

Proof of Theorem 1.2 The proof is an immediate consequence of Theorem 4.11. □

Remark 4.12 Let $\mathcal{C}$ be a braided fusion category such that $\mathrm{c.d.}(\mathcal{C}) = \{1, 2\}$. Suppose that $\mathcal{C}$ is nilpotent. By [4, Theorem 1.1] $\mathcal{C}$ admits a unique decomposition (up to the order of factors) into a tensor product $\mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_m$, where $\mathcal{C}_i$ are braided fusion categories of Frobenius-Perron dimension $p_i^{m_i}$, for some pairwise distinct prime numbers $p_1, \dots, p_m$. Then $\mathcal{C}_i$ is an integral braided fusion category, for all $i = 1, \dots, m$, and by [8, Theorem 2.11], we get that $\mathcal{C}_i$ is pointed whenever $p_i > 2$. Hence $\mathcal{C} \simeq \mathcal{C}_1 \boxtimes \mathcal{B}$ as braided fusion categories, where $\mathcal{C}_1$ is a braided fusion category of Frobenius-Perron dimension 2^m such that $\mathrm{c.d.}(\mathcal{C}_1) = \{1, 2\}$, and $\mathcal{B}$ is a pointed braided fusion category.

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